JACOB'S LADDERS AND SOME NONLINEAR INTEGRAL EQUATIONS CONNECTED WITH THE POISSON-LOBACHEVSKY INTEGRAL

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Abstract. We obtain some new properties of the signal generated by the Riemann zeta-function in this paper. Namely, we show the connection between the function $\zeta\left(\frac{1}{2}+it\right)$ and a nonlinear integral equation related to the Poisson-Lobachevsky integral.

1. The result

1.1. Let us remind the Poisson parametric integral

$$\int_0^{\pi}\ln(1+a^2-2a\cos x)\mathrm{d}x=\left\{\begin{array}{ccc} 0 & , & a\in(0,1), \\ 2\pi\ln a & , & a>1. \end{array}\right. \tag{1.1}$$

It is well-known that Lobachevsky applied his new geometric system to the calculations of the wide set of integrals. One of these integrals (see [2], eq. (111)) leads, after a small transformation, right to the integral (1.1).

Remark 1. The integral (111) of [2] was used by Lobachevsky to answer the unfair criticism of Ostrogradsky.

1.2. In this paper we obtain some new properties of the signal

$$Z(t)=e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right)$$

generated by the Riemann zeta-function, where

$$\vartheta(t)=-\frac{t}{2}\ln\pi+\mathrm{Im}\ln\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)=\frac{t}{2}\ln\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}+\mathcal{O}\left(\frac{1}{t}\right),$$

namely, the nonlinear integral equation connected with the function $\zeta\left(\frac{1}{2}+it\right)$ and the Poisson-Lobachevsky integral (1.1).

Next let us remind that

$$\tilde{Z}^2(t)=\frac{\mathrm{d}\varphi_1(t)}{\mathrm{d}t},\ \varphi_1(t)=\frac{1}{2}\varphi(t),$$

where

$$\tilde{Z}^2(t)=\frac{Z^2(t)}{2\Phi'_{\varphi}[\varphi(t)]}=\frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^2}{\left\{1+\mathcal{O}\left(\frac{\ln\ln t}{\ln t}\right)\right\}\ln t} \tag{1.2}$$

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(see [3], (3.9); [4] (1.3); [9], (1.1), (3.1), (3.2)), and φ is the Jacob's ladder, i.e. a solution of the nonlinear integral equation (see [3])

$$\int_0^{\mu[x(T)]} Z^2(t)e^{-\frac{2}{x(T)}t}\mathrm{d}t = \int_0^T Z^2(t)\mathrm{d}t,$$

(there is an infinite set of Jacob's ladders).

1.3. The following theorem holds true.

Theorem. Every Jacob's ladder $\varphi_1(t) = \frac{1}{2}\varphi(t)$, where $\varphi(t)$ is the exact solution of the nonlinear integral equation

$$\int_0^{\mu[x(T)]} Z^2(t)e^{-\frac{2}{x(T)}t}\mathrm{d}t = \int_0^T Z^2(t)\mathrm{d}t,$$

is the asymptotic solution of the following nonlinear integral equation

$$\frac{\int_{x^{-1}(T+\tau_a)}^{x^{-1}(T+\pi)} \ln[1+a^2-2a\cos(x(t)-T)]\left|\zeta\left(\frac{1}{2}+it\right)\right|^2}{-\int_{x^{-1}(T)}^{x^{-1}(T+\tau_a)} \ln[1+a^2-2a\cos(x(t)-T)]\left|\zeta\left(\frac{1}{2}+it\right)\right|^2} = 1, \tag{1.3}$$

where $x(t) = x(t;a)$, and

$$\tau_a = \arccos\frac{a}{2},\ a\in(0,1),$$

i.e. the following asymptotic formula

$$\frac{\int_{\varphi_1^{-1}(T+\tau_a)}^{\varphi_1^{-1}(T+\pi)} \ln[1+a^2-2a\cos(x(t)-T)]\left|\zeta\left(\frac{1}{2}+it\right)\right|^2}{-\int_{\varphi_1^{-1}(T)}^{\varphi_1^{-1}(T+\tau_a)} \ln[1+a^2-2a\cos(x(t)-T)]\left|\zeta\left(\frac{1}{2}+it\right)\right|^2} = 1 + \mathcal{O}\left(\frac{\ln\ln T}{\ln T}\right), \tag{1.4}$$

as $T\to\infty$ holds true for every Jacob's ladder and for every fixed $a\in(0,1)$.

Remark 2. There are the fixed point methods and other methods of the functional analysis used to study the nonlinear equations. What can be obtained by using these methods in the case of the nonlinear integral equation (1.3)?

This paper is a continuation of the series [3] - [22].

2. Corollaries and remarks

2.1. It is clear that (1.3) is followed by:

Corollary 1.

$$\int_{x^{-1}(T)}^{x^{-1}(T+\pi)} \ln[1+a^2-2a\cos(x(t)-T)]\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t = 0, \tag{2.1}$$

i.e. we have homogenous nonlinear integral equation.

Next, from (2.1), in the case $a=\frac{1}{b},\ b>1$ we obtain

Corollary 2.

$$\int_{x^{-1}(T)}^{x^{-1}(T+\pi)} \ln[1+b^2-2b\cos(x(t)-T)]\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t = $$
$$= 2\ln b\int_{x^{-1}(T)}^{x^{-1}(T+\pi)}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t,$$

(comp. with the integral (1.1)).

Remark 3. If we apply the usual way in the case (1.1), $a \geq 2$, we obtain the following theorem: every Jacob's ladder $\varphi_1(t) = \frac{1}{2}\varphi(t)$, where $\varphi(t)$ is the exact solution of the nonlinear integral equation

$$\int_0^{\mu[x(T)]} Z^2(t) e^{-\frac{2}{x(T)}t} \mathrm{d}t = \int_0^T Z^2(t)\mathrm{d}t,$$

is the asymptotic solution of the following nonlinear integral equation

$$\begin{aligned} & \int_{x^{-1}(T)}^{x^{-1}(T+\pi)} \ln[1 + a^2 - 2a\cos(x(t) - T)] \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \mathrm{d}t = \\ & = 2\pi \ln a \ln T,\ a \geq 2, \end{aligned}$$

where $\ln[1 + a^2 - 2a\cos(x(t) - T)] \geq 0$, i.e. the following asymptotic formula

$$\begin{aligned} & \int_{\varphi_1^{-1}(T)}^{\varphi_1^{-1}(T+\pi)} \ln[1 + a^2 - 2a\cos(\varphi_1(t) - T)] \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \mathrm{d}t \sim \\ & \sim 2\pi \ln a \ln T,\ T \to \infty \end{aligned} \tag{2.2}$$

holds true.

2.2. On Riemann hypothesis the following Littlewood's estimate

$$0 < \gamma' - \gamma < \frac{A}{\ln\ln\gamma},\ \gamma \to \infty$$

holds true (see [1]), where γ, γ' stand for consecutive zeroes of the function $\zeta\left(\frac{1}{2} + it\right)$. Since

$$\frac{1}{\gamma' - \gamma} > \frac{1}{A} \ln\ln\gamma > 2,\ \gamma > \gamma_0,$$

then we obtain from (2.1)

Corollary 3. On Riemann hypothesis

$$\begin{aligned} & \int_{\varphi_1^{-1}(\gamma)}^{\varphi_1^{-1}(\gamma+\pi)} \ln\left[1 + \frac{1}{(\gamma'-\gamma)^2} - \frac{2}{\gamma'-\gamma}\cos(\varphi_1(t) - \gamma)\right] \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \mathrm{d}t \sim \\ & \sim -2\pi \ln(\gamma' - \gamma)\ln\gamma,\ \gamma > \max\{\gamma_0, T_0[\varphi_1]\}. \end{aligned}$$

Next, we obtain

Corollary 4.

$$\begin{aligned} & \int_{\varphi_1^{-1}(p)}^{\varphi_1^{-1}(p+\pi)} \ln[1 + (p'-p)^2 - 2(p'-p)\cos(\varphi_1(t) - p)] \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \mathrm{d}t \sim \\ & \sim 2\pi \ln(p' - p)\ln p,\ p \geq T_0[\varphi_1], \end{aligned}$$

where $p, p';\ p < p'$ denote the consecutive prime numbers.

3. Proof of the Theorem

3.1. Let us remind that the following lemma holds true (see [8], (2.5); [9], (3.3)): for every integrable function (in the Lebesgue sense) $f(x),\ x\in[\varphi_1(T),\varphi_1(T+U)]$ we have

$$\int_T^{T+U} f[\varphi_1(t)]\tilde{Z}^2(t)\mathrm{d}t=\int_{\varphi_1(T)}^{\varphi_1(T+U)} f(x)\mathrm{d}x,\ U\in\left(0,\frac{T}{\ln T}\right] \tag{3.1}$$

where

$$t-\varphi_1(t)\sim(1-c)\pi(t), \tag{3.2}$$

c is the Euler's constant and $\pi(t)$ is the prime-counting function. In the case $\overset{\circ}{T}=\varphi_1^{-1}(T),\ \overset{\circ}{\widehat{T+U}}=\varphi_1^{-1}(T+U)$ we obtain from (3.1)

$$\int_{\varphi_1^{-1}(T)}^{\varphi_1^{-1}(T+U)} f[\varphi_1(t)]\tilde{Z}^2(t)\mathrm{d}t=\int_T^{T+U} f(x)\mathrm{d}x. \tag{3.3}$$

3.2. We obtain from (1.1), $0<a<1$

$$\int_{\tau_a}^{\pi}\ln(1+a^2-2a\cos\tau)\mathrm{d}\tau=-\int_0^{\tau_a}\ln(1+a^2-2a\cos\tau)\mathrm{d}\tau, \tag{3.4}$$

where

$$\tau_a=\arccos\frac{a}{2}, \tag{3.5}$$

and

$$\begin{aligned}&\ln(1+a^2-2a\cos\tau)>0,\ \tau\in(\tau_a,\pi),\\&\ln(1+a^2-2a\cos\tau)<0,\ \tau\in(0,\tau_a).\end{aligned} \tag{3.6}$$

Putting

$$f(t)=\ln[1+a^2-2a\cos(t-T)],\ U=\pi;\ t=\tau+T$$

in (3.3), we obtain (see (3.3), (3.4))

$$\begin{aligned}&\int_{\varphi_1^{-1}(T+\tau_a)}^{\varphi_1^{-1}(T+\pi)}\ln[1+a^2-2a\cos(\varphi_1(t)-T)]\tilde{Z}^2(t)\mathrm{d}t=\\&=-\int_{\varphi_1^{-1}(T)}^{\varphi_1^{-1}(T+\tau_a)}\ln[1+a^2-2a\cos(\varphi_1(t)-T)]\tilde{Z}^2(t)\mathrm{d}t\end{aligned} \tag{3.7}$$

where

$$\int_T^{T+\pi}\ln[1+a^2-2a\cos(t-T)]\mathrm{d}t=\int_0^{\pi}\ln(1+a^2-2a\cos\tau)\mathrm{d}\tau.$$

3.3. We obtain by using the mean-value theorem for the integrals in (3.7) (see (1.2), (3.6))

(3.8)

$$\int_{\varphi_1^{-1}(T+\tau_a)}^{\varphi_1^{-1}(T+\pi)} \ln[1+a^2-2a\cos(\varphi_1(t)-T)]\tilde{Z}^2(t)\mathrm{d}t =$$

$$= \frac{1}{\left\{1+\mathcal{O}\left(\frac{\ln\ln t_1}{\ln t_1}\right)\right\}\ln t_1}\int_{\varphi_1^{-1}(T+\tau_a)}^{\varphi_1^{-1}(T+\pi)} \ln[1+a^2-2a\cos(\varphi_1(t)-T)]\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t,$$

$$\int_{\varphi_1^{-1}(T)}^{\varphi_1^{-1}(T+\tau_a)} \ln[1+a^2-2a\cos(\varphi_1(t)-T)]\tilde{Z}^2(t)\mathrm{d}t =$$

$$= \frac{1}{\left\{1+\mathcal{O}\left(\frac{\ln\ln t_2}{\ln t_2}\right)\right\}\ln t_2}\int_{\varphi_1^{-1}(T)}^{\varphi_1^{-1}(T+\tau_a)} \ln[1+a^2-2a\cos(\varphi_1(t)-T)]\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t,$$

where $t_1, t_2 \in (\varphi_1^{-1}(T), \varphi_1^{-1}(T+\pi))$ and

$$t_1 = \varphi_1^{-1}(T_1),\ t_2 = \varphi_1^{-1}(T_2),\ T_1, T_2 \in (T, T+\pi). \tag{3.9}$$

3.4. Next, we obtain from (3.2) by (3.9) ($t_1 \to \infty \ \Leftrightarrow\ T \to \infty$)

$$t_1 - T_1 = \mathcal{O}\left(\frac{t_1}{\ln t_1}\right) \ \Rightarrow\ 1 - \frac{T_1}{t_1} = \mathcal{O}\left(\frac{1}{\ln t_1}\right) \to 0,\ T \to \infty, \tag{3.10}$$

i.e.

$$t_1 \sim T_1 \sim T,\ t_2 \sim T_2 \sim T, \tag{3.11}$$

and (see (3.10), (3.11); $0 < T_1 - T, T_2 - T < \pi$)

$$t_1 - T = \mathcal{O}\left(\frac{T}{\ln T}\right),\ t_2 - T = \mathcal{O}\left(\frac{T}{\ln T}\right). \tag{3.12}$$

Now,

$$\ln t_1 = \ln T + \mathcal{O}\left(\frac{t_1 - T}{T}\right) = \ln T + \mathcal{O}\left(\frac{1}{\ln T}\right), \tag{3.13}$$

and similarly,

$$\ln t_2 = lnT + \mathcal{O}\left(\frac{1}{\ln T}\right). \tag{3.14}$$

Then the formula (1.3) follows from (3.7) by (3.8), (3.13) and (3.14).

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